On the number of connected components in complements to arrangements of submanifolds

I. N. Shnurnikov*

Abstract

We consider arrangements of n connected codimensional one submanifolds in closed $d-$dimensional manifold M. Let f be the number of connected components of the complement in M to the union of submanifolds. We prove the sharp lower bound for f via n and homology group $H_{d-1}(M)$. The sets of all possible $f-$ values for given n are studied for hyperplane arrangements in real projective spaces and for subtori arrangements in $d-$ dimensional tori.

Introduction

The theory of plane arrangements in affine or projective spaces has been investigated rather thoroughly, see the book of P. Orlic, H. Terao, [4] and V. A. Vassiliev's review [8]. Inspired by a conjecture of B. Grünbaum [2], N. Martinov [3] found all possible pairs (n, f) such that there is a real projective plane arrangement of n pseudolines and f regions. It turns out, that some facts concerning arrangements of hyperplanes or oriented matroids could be generalized to arrangements of submanifolds, see P. Deshpande dissertation [1]. So we are going to study the sets $F(M, n)$ of connected components numbers of the complements in the closed manifold M to the unions of n closed connected codimensional one submanifolds. Sometimes it seems reasonable to restrict the type of submanifolds, for example, author [7] found sets $F(M, n)$ of region numbers in arrangements of n closed geodesics in the two dimensional torus and the Klein bottle with locally flat metrics.

Homological bound of the number of connected components

Let M^n be connected n–dimensional smooth compact manifold without boundary, let $A_i \subset M^n$ be distinct connected $(n-1)$–dimensional closed submanifolds in M^n for $1 \le i \le k$. Let us consider the union

$$A = \bigcup_{i=1}^{k} A_i.$$

We shall denote by f the number $|\pi_0(M^n \setminus A)|$ of connected components of the complement to A in M^n. Let UA be regular open neighbourhood of A in M^n. Let

$$M^n \setminus UA \cong \bigsqcup_{j=1}^{f} N_j,$$

where N_j are the connected components of the complement to UA in M^n. If M^n and all submanifolds A_i are orientable, then we assume $G = \mathbb{Z}$. If some A_i or M^n is not orientable, then $G = \mathbb{Z}_2$.

*Yaroslavl State University, Delone Laboratory of Discrete and Computational Geometry. This work is supported by the Russian Government project 11.G34.31.0053.

Lemma 1. *If closed $(n-1)$ – dimensional submanifolds $A_i \subset M^n$, $i = 1, \ldots, k$ intersect each other transversally, then*

$$\dim H_{n-1}(UA, G) = \dim H_{n-1}(A, G) \geq k$$

Proof. The regular neighbourhood of UA is homotopically equivalent to A and so all homology groups of A and UA are the same. By induction on k let us prove

$$\dim H_{n-1}\left(\cup_{i=1}^{k} A_i, G\right) \geq k.$$

It is obvious for $k = 1$ because for connected closed $(n-1)$ – dimensional manifold $H_{n-1}(A_1, G) \cong G$. Suppose the statement is true for $k-1$ submanifolds and let us prove it for k submanifolds. Let

$$A' = \bigcup_{i=1}^{k-1} A_i.$$

Then by induction assumption

$$\dim H_{n-1}(A', G) \geq k - 1.$$

By Meyer-Vietoris exact sequence for pair A', A_k we have:

$$\longrightarrow H_{n-1}(A' \cap A_k) \longrightarrow H_{n-1}(A') \oplus H_{n-1}(A_k) \longrightarrow H_{n-1}(A' \cup A_k) \longrightarrow$$

As submanifolds A_i and A_j intersect transversally then $A' \cap A_k$ is a finite union of at most $(n-2)$ – dimensional submanifolds in M^n. Hence $H_{n-1}(A' \cap A_k) = 0$ and the map

$$H_{n-1}(A') \oplus H_{n-1}(A_k) \longrightarrow H_{n-1}(A' \cup A_k)$$

is monomorphic. Therefore,

$$\dim H_{n-1}(A' \cup A_k) \geq \dim H_{n-1}(A') + \dim H_{n-1}(A_k) \;\geq\; k.$$

□

Lemma 2.

$$H_n(M^n, UA, G) \cong G^f$$

Proof.

$$\begin{aligned} H_n(M^n, UA, G) &= \widetilde{H}_n(M^n/UA, G) = \\ &= \widetilde{H}_n\left(\sqcup_{j=1}^{f} N_j / \sqcup_{j=1}^{f} \partial N_j,\ G\right) = \widetilde{H}_n\left(\vee_{j=1}^{f} N_j/\partial N_j, G\right) = \\ &= \bigoplus_{j=1}^{f} \widetilde{H}_n(N_j/\partial N_j, G) = G^f, \end{aligned}$$

where $n \geq 1$, $\vee$ is one point union, $\widetilde{H}_n$ is the reduced homology group, ∂N_j is the boundary of N_j. □

Theorem 1. *Let $A_1, \ldots, A_k$ be connected closed codimensional one submanifolds in a connected closed n – dimensional manifold M^n. Suppose that the submanifolds A_i intersect each other transversally and $A = \cup_i A_i$. Then*

$$|\pi_0(M^n \setminus A)| \geq k + 1 - \dim H_{n-1}(M^n, G),$$

where G is chosen as before.

Proof. Let us write the exact homological pair sequence for inclusion $i : UA \to M^n$ with coefficients in G:

$$H_n(UA) \longrightarrow H_n(M^n) \longrightarrow H_n(M^n, UA) \longrightarrow H_{n-1}(UA) \longrightarrow H_{n-1}(M^n) \longrightarrow$$

Notice that

$$H_n(UA) = H_n(A) = 0, \quad H_n(M_n) = G.$$

It follows from the exactness of sequence in $H_n(M^n)$, that the map

$$H_n(M^n) \longrightarrow H_n(M^n, UA)$$

is monomorphic. By lemma 2

$$H_n(M^n, UA, G) \cong G^f$$

One can see that

$$\dim H_{n-1}(UA) \leq \dim \mathrm{Im}\partial_* + \dim \mathrm{Im} i_*,$$

where the homomorphisms are

$$\partial_* : H_n(M^n, UA) \longrightarrow H_{n-1}(UA), \qquad i_* : H_{n-1}(UA) \longrightarrow H_{n-1}(M^n).$$

Notice that

$$\dim \mathrm{Im} i_* \leq \dim H_{n-1}(M^n), \quad \dim \mathrm{Im}\partial_* \leq f - 1.$$

By lemma 1 $\dim H_{n-1}(UA) \geq k$ and so $k \leq f - 1 + \dim H_{n-1}(M^n)$. □

Remark 1. One can see that the inequality of the theorem is sharp for arrangements of

$$n \geq \dim H_{n-1}(M^n)$$

submanifolds in projective spaces, spheres, n – dimensional tori and Riemann surfaces of genus g.

Toric arrangements

Definition 1. *By a flat d – dimensional torus T^d we mean a quotient of affine d – dimensional space by a nondegenerate d – lattice Z^d (which is not surely integer lattice). A codimensional one subtorus is given by equation*

$$\sum_i a_i x_i = c,$$

where a_i are rational, x_i are coordinates of $\mathbb{R}^d$ in some lattice basis, c is real.

A codimensional one subtorus is closed submanifold of T^d homeomorphic to $(d-1)$ – dimensional torus. Let A be the union of n codimensional one subtori in the flat d – dimensional torus T^d. Consider the connected components of the complement $T^d \setminus A$; denote the number of connected components by $f = |\pi_0\left(T^d \setminus A\right)|$; let $F\left(T^d, n\right)$ be the set of all possible numbers f.

Theorem 2. *For $n > d$*

$$F\left(T^d, n\right) \supseteq \{n - d + 1, \ldots, n\} \cup \{l \in \mathbb{N} \mid l \geq 2(n - d).\}$$

For $2 \leq n \leq d$ we have $F\left(T^d, n\right) = \mathbb{N}$.

Proof. Let $T^d = \mathbb{R}^d / Z^d$ and e be the basis of Z^d. Let $(x_1, \dots, x_d)$ be the coordinates of $\mathbb{R}^d$ in the basis e. We shall construct examples for $\leq n$ and $\geq 2n - 2d$ regions separately.

Let us consider n hyperplanes in $\mathbb{R}^d$ (an equation corresponds to a hyperplane):

$$x_i = 0, \quad 1 \leq i \leq k,$$
$$x_{k+1} = c_{i-k}, \quad k+1 \leq i \leq n$$

for some integer k, $0 \leq k \leq d-1$ and real c_{i-k} with different frctional parts. By the factorization map $\mathbb{R}^d \to \mathbb{R}^d / Z^d$ we shall get a set $\left\{T_i^{d-1}, i = 1, \dots, n\right\}$ of n codimensional one subtori. And the complement is homeomorphic to the prime product

$$T^d \setminus \bigcup_i T_i^{d-1} \approx \mathbb{R}^k \times \left(S^1 \setminus \{p_1, \dots, p_{n-k}\}\right) \times \left(S^1\right)^{d-k-1},$$

where $S^1 \setminus \{p_1, \dots, p_{n-k}\}$ denotes a circle without $n-k$ points. Hence the number of complement regions equals $n-k$, for an integer k such that $0 \leq k \leq d-1$.

Now let us take integer nonnegative k and construct an arrangement with $2n - 2d + k$ connected components of the complement. We shall determine the subtori by equations:

$$x_i = 0, \quad \text{for} \quad 2 \leq i \leq d,$$
$$x_2 = kx_1 + \frac{1}{2},$$
$$x_1 = c_j \quad \text{for} \quad j = 1, \dots, n-d,$$

whereas numbers $kc_j + \frac{1}{2}$ are not integer for any j. (This means that the intersection of three subtori

$$x_2 = kx_1 + \frac{1}{2}, \ x_1 = c_j, \ x_2 = 0$$

is an empty set.) One may see that

$$T^d \setminus \bigcup_{i=3}^{d} \{x_i = 0\} \approx T^2 \times \mathbb{R}^{d-2}.$$

In the two$-$dimensional torus the equations

$$x_2 = 0,$$
$$x_2 = kx_1 + \frac{1}{2},$$
$$x_1 = c_j \text{ for } j = 1, \dots, n-d$$

produce the arrangement of $n - d + 2$ closed geodesics. The geodesics' union divides the torus into $2n - 2d + k$ connected components (for more details on arrangements of closed geodesics in the flat torus see author's paper [7]). □

Conjecture 1. *It seems believable that the inclusion in the theorem is indeed the equality for all $d \geq 2$ and $n \geq d$. Yet the equality is proved for $d = 2$ in [7].*

Sets of region's numbers in hyperplane arrangements

By an arrangement of n hyperplanes in the real projective space $\mathbb{RP}^d$ we mean a set of n hyperplanes, such that there are no point belonging to all the hyperplanes. The arrangement produce the cell decomposition of the $\mathbb{RP}^d$; let f denotes the number of open $d-$cells. Let $F_n^{(d)}$ denotes the set of all possible numbers f arising in arrangements of n hyperplanes in $\mathbb{RP}^d$. Let m be the maximal number of hyperplanes, passing through one point.

Lemma 3. *For arrangements of n hyperplanes in $\mathbb{RP}^d$ we have*

$$f \geq (m-d+1)\sum_{j=0}^{[\frac{d}{2}]} \frac{C_n^{d-2j}}{C_{m-2j}^{d-2j}}.$$

Proof. It follows from Zaslavsky formula for number of regions and some inequalities concerning the Möbius function of the arrangement poset. □

Lemma 4. *For arrangement of n hyperplanes in the real projective space $\mathbb{RP}^d$*

$$f \geq (n-m+1)(m-d+2)2^{d-2}.$$

Proof. Let m hyperplanes $A_1, \ldots, A_m$ have nonempty intersection Q (Q is a point). The family $A_1, \ldots, A_m$ is a cone over some arrangement B of m planes in $\mathbb{RP}^{d-1}$. The number $f(B)$ of regions in arrangement B could be estimated (see Shannon paper [5], where this result is referred to McMullen) as:

$$f(B) \geq (m-d+2)2^{d-2}.$$

Each of the remaining hyperplane of the former arrangement intersects the family $A_1, \ldots, A_k$ by an arrangement B_i, projective equivalent to B. Thus

$$f \geq f(B) + \sum_i f(B_i) = (n-m+1)f(B).$$

□

Theorem 3. *Let $d \geq 3$ and $n \geq 2d+5$. Then the first four increasing numbers of $F_n^{(d)}$ are the following:*

$$(n-d+1)2^{d-1}, \quad 3(n-d)2^{d-2}, \quad (3n-3d+1)2^{d-2}, \quad 7(n-d)2^{d-3}.$$

Proof. We are going to prove that the four mentioned numbers are the only realizable ones among numbers not greater then $7(n-d)2^{d-3}$. After it one may see how to construct examples of arrangements with required numbers f. Let us prove that if $m \leq d+1$, then

$$f \geq 7(n-d)2^{d-3}.$$

For $m = d$ we have an arrangement of hyperplanes in general position and the number of regions is the largest possible. If $m = d+1$, then by lemma 3 we have

$$f \geq \frac{C_n^{d+1}}{n-d} = \frac{n}{3}\frac{(n-1)}{d+1}\frac{(n-2)}{d}\frac{(n-3)}{d-1}\cdots\frac{(n-d+2)}{4}(n-d+1) \geq 7 \cdot 2^{d-3}(n-d)$$

because $n \geq 2d+5$.

Now we prove the theorem for $d = 3$, $n \geq 11$. Let us consider three cases.

1. If $m = n-1$, then $f = 2\varphi$, where $\varphi \in F_{n-1}^{(2)}$. The set $F_{n-1}^{(2)}$ is known due to N. Martinov [3]

$$\{f \in F_{n-1}^{(2)} \mid f \leq 4n-16\} = \{2n-4, 3n-9, 3n-8, 4n-16\}.$$

2. $m = n-2$. The arguments are the same as in the inductive step further (Martinov theorem [3] for the set $F_n^{(2)}$ is also used).

3. If $5 \leq m \leq n-3$, then by using lemma 4 we have

$$f \geq 2(n-m+1)(m-1) \geq 8n-32 \geq 7n-21$$

for $n \geq 11$.

Now we use induction on $d \geq 3$. Base is the validity of the theorem for $d = 3$. The assumption is the validity of the theorem for all integers $3 \leq d' < d$ and $n' \geq 2d' + 5$. To prove the induction step we shall consider three cases.

1. If $m = n - 1$, then $f = 2\varphi$, where $\varphi \in F_{n-1}^{(d-1)}$. By induction assumption for the set $F_{n-1}^{(d-1)}$ (note that $n - 1 \geq 2(d - 1) + 1$) we get that either φ is equal to one of four numbers

$$(n-d+1)2^{d-2}, \quad 3(n-d)2^{d-3}, \quad (3n-3d+1)2^{d-3}, \quad 7(n-d)2^{d-4},$$

or $\varphi > 7(n-d)2^{d-4}$.

2. $m = n - 2$. Consider $n - 2$ hyperplanes $p_1, \dots, p_{n-2}$, passing through one point. These hyperplanes cut $\mathbb{RP}^d$ into φ regions and $\varphi \in F_{n-2}^{(d-1)}$. Let l denote the intersection of the two remaining hyperplanes. By the inductive assumption we have either

$$\varphi = (n-d)2^{d-2} \quad \text{or} \quad \varphi \geq 3(n-d-1)2^{d-3}$$

(note that assumption may be used as $n - 2 \geq 2(d - 1) + 5$). If

$$l \in \bigcup_{i=1}^{n-2} p_i$$

then $f = 3\varphi$ and the case is over. If

$$l \notin \bigcup_{i=1}^{n-2} p_i$$

then let B be the set of planes $p_i \cap l$ in the l, where l is regarded as the ambient $(d-2)-$dimensional projective space. One may prove, that B is an arrangement of at least $n - 3$ planes in l. Then $f(B) \geq (n-d)2^{d-3}$ by Shannon theorem [5]. Since

$$f = 3\varphi + f(B) \geq 7(n-d)2d - 3,$$

the case is over.

3. If $d + 2 \leq m \leq n - 3$ then by lemma 4 we have

$$f \geq (n-m+1)(m-d+2)2^{d-2} \geq (4n-4d-4)2^{d-2} \geq 7(n-d)2^{d-3}$$

for $n \geq d + 8$. □

Lemma 5. *For arrangement of n hyperplanes in the real projective space* $\mathbb{RP}^d$

$$f \geq 2\frac{n^2 - n}{m - d + 5}.$$

Proof. It follows from the similar inequality for arrangement of lines in the projective plane, see details in [6]. □

Theorem 4. *First 36 increasing numbers of the set* $F_n^{(3)}$ *for* $n \geq 50$ *are the following (i.e. all realizable numbers up to* $12n - 60$*)*

$$4n-8,\ 6n-18,\ 6n-16,\ 7n-21,\ 7n-20,\ 8n-32,\ 8n-30,\ 8n-28,$$
$$8n-26,\ 9n-36,\ 9n-33,\ 9n-31,\ 9n-30,\ 10n-50,\ 10n-48,\ 10n-46,$$
$$10n-44,\ 10n-42,\ 10n-40,\ 10n-39,\ 10n-38,\ 10n-37,\ 10n-36,\ 10n-35,$$
$$11n-44,\ 11n-43,\ 11n-42,\ 11n-41,\ 11n-40,\ 12n-72,\ 12n-70,\ 12n-68,$$
$$12n-66,\ 12n-64,\ 12n-62,\ 12n-60.$$

Proof. Let m be the maximal number of hyperplanes, passing through one point. Examples for this numbers could be constructed for arrangements with $m \geq n-5$. Let us prove that there are no other realizable numbers, smaller then $12n-60$. Consider three cases.

1. It $m \geq n-5$, then by enumeration of possibilities we have that either f belongs to given set or $f \geq 12n-60$.

2. If $8 \leq m \leq n-6$, then by lemma 4 we have $f \geq 7n-49$.

3. If $m \leq 7$ then by lemma 5

$$f \geq 2\frac{n^2-n}{9} \geq 12n-60$$

for $n \geq 50$. □

References

[1] P. Deshpande, Arrangements of Submanifolds and the Tangent Bundle Complement. *Electronic Thesis and Dissertation Repository*, Paper 154 (2011).

[2] B. Grünbaum, *Arrangements and Spreads.* AMS, Providence, Rhode Island, 1972.

[3] N. Martinov, Classification of arrangements by the number of their cells. *Discrete and Comput. Geometry* (1993) **9**, N 1, 39–46.

[4] P. Orlic, H. Terao, Arrangements of Hyperplanes. Springer – Verlag, Berlin – Heidelberg, 1992. 329 pp.

[5] R.W. Shannon, A lower bound on the number of cells in arrangements of hyperplanes. *Jour. of combinatorial theory (A)*, **20**, (1976) 327–335.

[6] I. N. Shnurnikov, Into how many regions do n lines divide the plane if at most $n-k$ of them are concurrent? *Moscow Univ. Math. Bull., ser. 1* (2010) **65**:5, 208 – 212.

[7] I. N. Shnurnikov, On the number of regions formed by arrangements of closed geodesics on flat surfaces, *Math. Notes* **90**, N 3 – 4 (2011), 619 – 622.

[8] V. A. Vassiliev, Topology of plane arrangements and their complements. *Uspekhi Mat. Nauk*, **56**, iss. 2(338), (2001), 167 — 203.